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COMMENT

# The Edwards model and the weakly self-avoiding walk

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**Abstract.** It is commented that the fact that the Edwards model has paths of Hausdorff dimension two does not contradict the conjecture that the exponent for root mean square distance for random walks should equal the reciprocal of the dimension of the sample paths.

The recent article of Koukiou, Pasche and Petritis (1989) notes that the paths of the Edwards model in two dimensions have Hausdorff dimension two almost surely. This fact follows immediately from the fact that the Edwards model is absolutely continuous with respect to Wiener measure. It is then stated that this gives a counterexample to the conjecture that the 'dimension' $\bar{d}$ of paths is related to the root mean square distance exponent $\nu$ by $\bar{d} = 1/\nu$. In the case of the weakly self-avoiding walk in two dimensions it is believed that $\nu = 3/4$. What we wish to note here is that this is not a counterexample to the conjecture if the conjecture is stated precisely.

It will be necessary to define rigorously what is meant by the 'dimension' of paths. We will use Hausdorff dimension in the strict mathematical sense. Therefore in order to take the 'dimension' of lattice paths, we must first go to the continuum limit and then consider the Hausdorff paths of the limit process (strictly speaking, the Hausdorff dimension of any lattice random walk path before going to the continuum limit is one). Let $\Lambda_n$ be the set of nearest-neighbour random walks $\omega(j)$, $0 \leqslant j \leqslant n$, of length $n$ starting at the origin in $Z^d$. Suppose $P_n$ is a sequence of probability measures on $\Lambda_n$. Let $a_n = a_n(P_n)$ be the root mean square displacement, $a_n = (\langle|\omega(n)|^2\rangle_{P_n})^{1/2}$. The exponent $\nu$ is defined by $a_n \approx n^\nu$. Let $\mu_n$ be the measure on $C[0, 1]$, the continuous functions from $[0, 1]$ to $R^d$, which assigns measure $P_n(\omega)$ to the function $f(\cdot)$ where

$$f(t) = \frac{\omega(j)}{a_n} \qquad \text{if } t = \frac{j}{n} \text{ for some } j$$

and is defined by linear interpolation for other $t \in [0, 1]$. Then we say $\mu$ is a continuum limit of $P_n$ if there exists a subsequence $n_i \to \infty$ with $\mu_{n_i} \to \mu$ weakly. We say the paths (or more precisely the sequence of measures $P_n$) have dimension $\bar{d}$ if there exists a continuum limit of the $P_n$ which gives measure one to the set of paths of Hausdorff dimension $\bar{d}$. If $P_n$ is the uniform measure on $\Lambda_n$, i.e. simple random walk, then $a_n = \sqrt{n}$ and the only continuum limit is Brownian motion.

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In defining the continuum limit for the weakly self-avoiding walk or the Edwards model one must consider two parameters: the interaction strength $\beta$ and the step size $n$. The weakly self-avoiding random walk is achieved by choosing $P_n$ in the following fashion: Let $I_n(\omega)$ be the number of self-intersections of the paths up through time $n$,

$$I_n(\omega) = \sum_{j=0}^{n} \sum_{k=0}^{n} \delta(\omega(j) - \omega(k)).$$

Let $\beta > 0$ and define

$$P_n(\omega) = \frac{\exp(-\beta I_n)}{\langle \exp(-\beta I_n) \rangle}.$$

We do not let $\beta$ depend on $n$. If there exists $n_i \to \infty$ and $\mu$ on $C[0, 1]$ such that $\mu_{n_i} \to \mu$ weakly, then $\mu$ is a continuum limit of the weakly self-avoiding walk. It is not known whether or not such a continuum limit exists. However, it is believed that $a_n \approx n^{3/4}$, and it is quite possible that the paths of any continuum limit have Hausdorff dimension $\frac{4}{3}$.

The Edwards model (Edwards 1965) is generally constructed by considering a sequence of measures defined on $C[0, 1]$. Let $B(t)$ be a Brownian motion and $W$ the corresponding Wiener measure on $C[0, 1]$. We formally define the measure $\mu$ by

$$\mathrm{d}\mu = \exp(-\beta J)\, \mathrm{d}W$$

where $J$ is the interaction

$$J = \int_0^1 \int_0^1 \delta(B(t) - B(s))\, \mathrm{d}s\, \mathrm{d}t.$$

Rigorous sense can be made of this if $d < 4$ by approximating the delta function and taking weak limits (Varadhan 1969, Westwater 1980). This measure is not a continuum limit of weakly self-avoiding walks in the sense of the previous paragraph. To see this it is easiest to take a discrete approximation. Fix an integer $n$. When we approximate a Brownian motion by a random walk of length $n$, the step size is $1/a_n = n^{-1/2}$. We therefore take as our approximate delta function,

$$\delta_n(x) = \begin{cases} n^{d/2} & x \in R_n \\ 0 & \text{otherwise} \end{cases}$$

where $R_n = \{(x_1, \ldots, x_d) : |x_i| \leq \frac{1}{2} n^{-1/2}\}$. When we approximate the Brownian motion by the scaled random walk

$$B_n(t) = \frac{\omega(j)}{\sqrt{n}} \qquad \text{if } t = \frac{j}{n}$$

then the interaction $J$ is approximated by

$$\begin{aligned} J_n &= \int_0^1 \int_0^1 \delta_n[B_n(s) - B_n(t)]\, \mathrm{d}s\, \mathrm{d}t \\ &\approx n^{-2} \sum_{j=0}^{n} \sum_{k=0}^{n} \delta\left[ B_n\left(\frac{j}{n}\right) - B_n\left(\frac{k}{n}\right)\right] \\ &= n^{(d/2)-2} I_n. \end{aligned}$$

For $d<4$, this interaction is significantly weaker than the interaction of the weakly self-avoiding walk. We call the measure

$$Q_n=\frac{\exp(-\beta n^{(d/2)-2}I_n)}{\langle\exp(-\beta n^{(d/2)-2}I_n)\rangle}$$

the *discrete Edwards model.* The terminology is reasonable since once can show that the continuum limit of the discrete Edwards model in two dimensions is the usual Edwards model; see e.g. Stoll (1985). The exponent for the root mean square distance of the discrete Edwards model should really be considered to be $\nu=\frac{1}{2}$ since the root mean square distance of paths under $Q_n$ is $\frac{1}{2}$. We see that this agrees with the conjecture.

For $d=1$, $\langle I_n\rangle\sim cn^{3/2}$, so the discrete Edwards model gives a measure that is 'absolutely continuous' with respect to Wiener measure. For $d=2$. $\langle I_n\rangle\sim cn(\log n)$, so it is not immediately clear that the measure is absolutely continuous. However, the interaction is quite uniform on paths; essentially, $I_n(\omega)=cn\log n+nK_n(\omega)$ where $K_n$ has bounded expectation and variance. This allows for the renormalisation first given by Varadhan (1969). The interaction is highly non-trivial in $d=3$; however, it is expected that the weak limit of the discrete Edwards model will be the same as the process constructed by Westwater (1980). This process is singular with respect to Wiener measure, but again is not a continuum limit of weakly self-avoiding walks. The discrete Edwards model is the same as the weakly self-avoiding walk in four dimensions, where it is believed that the only continuum limit is Brownian motion.

We have discussed the above by writing $\beta$ as a function of $n$. We can see a similar phenomenon if we write $n$ as a function of $\beta$. In this case if $n$ is of order $\beta^{-1}$ (for $d=2$) one gets behaviour like the Edwards model, while for $n$ much larger than $\beta^{-1}$ one gets behaviour like the weakly self-avoiding walk. In this sense the Edwards model and the weakly self-avoiding walk can be considered as the same process viewed at different times. Another way of obtaining the weakly self-avoiding walk as a limit of the Edwards model is to start with the original (continuous) Edwards model, and let time become large. Let $J^T$ be the interaction

$$J^T=\int_0^T\int_0^T\delta[B(s)-B(t)]\,\mathrm{d}s\,\mathrm{d}t$$

and let $\nu^T$ be the measure on $C[0,T]$ given by

$$\mathrm{d}\nu^T=\exp(-\beta J^T)\,\mathrm{d}W^T$$

where $W^T$ is the Wiener measure on $C[0,T]$. (Again we can define this precisely by taking approximate delta functions and weak limits.) Now define $\mu_T$ to be the measure on $C[0,1]$ induced from scaling $\nu_T$, i.e. let $\mu_T$ be the measure generated from $\nu_T$ and

$$B_T^*(t)=\frac{B(tT)}{\sqrt{\langle|B(T)|^2\rangle_{\nu_n}}}.$$

Then a weak limit of the measures $\mu_T$ could well be a continuum limit for weak self-avoiding walks. It should be noted that if $d=2$ paths have Hausdorff dimension two under the measures $\mu_T$, but there is no reason to believe that they would have Hausdorff dimension two in a weak limit of these measures (Hausdorff dimension is not preserved under weak limits; e.g., simple random walk paths have Hausdorff dimension one while the continuum limit, Brownian motion, has paths of Hausdorff dimension two).

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